Study Guide for Benchmark 3: Limits, Derivatives, and Integrals
Given in class on [Day 40]
Window-of-Opportunity: [Days 40 – 43]

The window-of-opportunity for Benchmark #3 is [Days 40 – 43]. This benchmark test will contribute 5% to your final course grade. There will be ten problems on calculating limits, derivatives, antiderivatives, and definite integrals. It will be offered during the last 30 minutes of class on [Day 40].

To pass this benchmark, you must get at least nine (of ten) problems completely correct; there are no partial credits. If you pass on your first attempt, your score will be recorded as 100%. If you do not pass on your first attempt, I will initially record your score as 0%, and you may retake this benchmark once or twice up through 4:00 pm on Monday, May 10. If you pass on a retest, your score will be changed to the average score of all your attempts. If you have not passed this benchmark by May 10 or after three attempts, your score will be recorded as the average of your three scores plus 0%. If you do not pass and do not attempt to retake the benchmark by May 10, the 0% that was initially recorded will stand.

In addition to the sample benchmark questions given below, there are six Web Work problem sets to assist in your preparation for this benchmark (B3 A – F). Each problem set includes a mixture of the kinds of problems that will appear on Benchmark #3 – limits, derivatives, antiderivatives, and definite integrals. All six problem sets will open on Monday, April 26.

- Web Work problem sets B3_A and B3_B will close at 5:00 pm on Monday, May 3;
- Set B3_C will close at 5:00 pm on Wednesday, May 5;
- Set B3_D will close at 5:00 pm on Friday, May 7;
- Sets B3_E and B3_F will close at 5:00 pm on Monday, May 10.

As each problem set closes, its answers will become available.

All students are required to complete problem sets B3_A and B3_B. To get credit for these problem sets, you must get at least 80% of the points available. That is, since each problem set is worth 5 points, you must earn 4 or more points per problem set. You may attempt each problem an unlimited number of times, but if you don't get the correct answer in about three attempts you are encouraged to ask me about the problem.

Those who pass the Benchmark on May 3 will be exempted from the remaining problem sets; those who do not pass the Benchmark on May 3, will be required to complete additional Web Work problem sets before retaking the Benchmark.

Topics to be covered by this Benchmark

Evaluating Limits

We need to calculate limits in situations where there is some problem with expression, such as a zero in the denominator, when the function makes a "jump" at a particular value, when a function has a vertical asymptote at a particular point, or when we are considering the ultimate value of an expression as the variable approaches positive or negative infinity. *(If you haven't already completed them, Web Works sets Ch10_limits A through E provide additional practice problem for calculating limits.)*

- If there is a "zero in the denominator" you can often fix the problem algebraically. If you can eliminate the zero in the denominator, you have an expression that is identical to the original expression – except at the problem point.
- A sketch graph of the function will often help us to "see" what is going on in the neighborhood of the problem points. You should be able to calculate limits by using a sketch graph.
- In order to compute the limit of an expression as the variable goes to infinity, it often helps to think about what is happening as the variable gets larger (goes toward $+\infty$) or gets smaller (goes toward $-\infty$). If the function is represented as a rational function, it is often sufficient to consider the highest power of the expression in the numerator divided by the highest power of the expression in the denominator.
- You should be able to calculate limits for constant functions, linear functions, polynomial functions, absolute value functions, and rational functions.

Calculating Derivatives
You should be able to calculate the following kinds of derivatives:
- Constant function, linear function, power function (x^p, for an integer or rational number p)
- Polynomial function
- Each of the six trigonometric functions
- Exponential function (a^x, $a^{u(x)}$, e^x, $e^{u(x)}$)
- Functions defined as a product or quotient
- Composite functions which require the chain rule

Calculating Antiderivatives and Definite Integrals
You should be able to calculate the following kinds of antiderivatives and definite integrals:
- Constant function, linear function
- Power function (x^p, for an integer or rational number p – for both $p \neq -1$, and for $p = -1$)
- Polynomial function
- The derivatives of each of the six trigonometric functions
- Exponential function (a^x, $a^{u(x)}$, e^x, $e^{u(x)}$)
- Functions defined as simple rational functions (for which you can set up a partial fraction decomposition)
- Functions requiring simple u-substitutions; that is, functions whose derivative was found using the chain rule
- Functions requiring integration by parts (u, dv substitution); that is, functions whose derivative was found using the chain rule

Sample Benchmark Questions
Evaluating limits (There will be 2 – 3 limit problems on this benchmark.)

1. If $y = 1/3x^2$, find the limit of y as $x \to \infty$.
2. If $y = e^{-2x}$, find the limit of y as $x \to \infty$.
3. If $y = e^{-2x}$, find the limit of y as $x \to -\infty$.
4. If $y = 1/3x$, find the limit of y as $x \to 0^-$
5. If $y = 1/3x$, find the limit of y as $x \to 0^+$
6. If $y = 1/3x$, find the limit of y as $x \to 0$
7. If $y = 1/5x^2$, find the limit of y as $x \to 0$
8. If $y = e^{-2x}$, find the limit of y as $x \to 0$
9. Calculate the limit: $\lim_{x\to\infty} \dfrac{5x^2 + 3x}{6 - 3x^2}$.
10. Calculate the limit: $\lim_{x\to 0} \dfrac{5x^2 + 3x}{6 - 3x^2}$.
11. Calculate the limit: $\lim_{x\to\infty} 5e^{-3x}$.
12. Calculate the limit: $\lim_{x\to 0} 5e^{-3x}$.
13. Calculate the limit: $\lim_{x\to 3} \dfrac{5x^2}{(x-3)(x+3)}$. Does the graph of this function have a vertical asymptote, a jump discontinuity, or a "hole" at the point where x = 3? Explain.

14. Calculate the limit: $\lim_{x \to 3} \frac{5x^2(x-3)}{x^2-9}$. Does the graph of this function have a vertical asymptote, a jump discontinuity, or a "hole" at the point where x = 3? Explain.
15. Calculate the limit: $\lim_{x \to 3} \frac{(x-3)(x+5)}{x^2-9}$. Does the graph of this function have a vertical asymptote, a jump discontinuity, or a "hole" at the point where x = 3? Explain.

Calculating derivatives (There will be 3 – 4 problems about derivatives on the benchmark.)

16. Find $f'(t)$ if $f(t) = 3t^2 + \sqrt{t} - 5/t$.
17. Find the derivative of $x^2 \cos(x)$.
18. Find the derivative of $R(t) = 3t + e^{3t} - \pi$.
19. Evaluate $f(x) = e^{-5x}$ at the point where x = 0.
20. Evaluate the derivative of $f(x) = e^{-5x}$ at the point where x = 0.
21. Find the slope of the tangent line to the graph of $x^2 \cos(x)$ at the point where $x = \pi/3$.
22. Find the equation of the tangent line to the graph of $x^2 \cos(x)$ at the point where $x = \pi/3$.
23. If $g(x) = 3x^2 \exp(x)$, find dg/dx.
24. At what points (t, r(t)) will the derivative of $r(t) = 4t^3 + 3t^2$ be equal to 0?
25. For what values of y will the derivative of $f(y) = 5y^3 - 3y^2 + 6y - 4$ be equal to 0?
26. Find the slope of the tangent line to the graph of $f(t) = 3t^2 - 5t$ at the point where *t = -1*.
27. Find the slope of the tangent line to the graph of $f(x) = \frac{1}{3x}$ at the point where *x = 5*.
28. If $f(x) = x^3 - x^2 - 6x$, find $\frac{df}{dx}$.
29. Find the derivative of $f(x) = \frac{x^2}{\sin(3x)}$.
30. Evaluate the derivative of $r(\theta) = \frac{\sin(\theta)}{(\theta^2 + 3)}$ at the point where θ = 0.

Calculating antiderivatives and definite integrals (There will be 3 – 4 problems about antiderivatives and definite integrals on this benchmark.)

31. Find the antiderivative of $y(x) = 5x^2 + \pi x + 3$.
32. If $y(x) = 5x^2 + \pi x + 3$, evaluate $\int_{-2}^{0} y(x)dx$.
33. Find the antiderivative of $R(t) = 3t + e^{3t} - \pi$.
34. Evaluate the definite integral: $\int_{1}^{5} 5x^2 + \pi x + 3$.
35. Find the antiderivative: $f(x) = x\cos(x)$.

36. Evaluate $\int_0^{\pi} x\cos(x)dx.$
37. Find the antiderivative: $g(y) = y\cos(y^2).$
38. Evaluate $\int_0^{\pi/2} y\sin(y^2)dy$
39. Find the antiderivative: $\int \frac{\sec^2(4x)}{\tan(4x)} = \int \frac{(\sec(4x))^2}{\tan(4x)}.$
40. Find int(y, x), if y(x) = 5*x^2 + sin(x) + 3.
41. Evaluate int(y, x=-Pi..Pi), if y(x) = 5*x^2 + sin(x) + 3.
42. Find int(a, t), if a(t) = 5/(t^2) + 3/t.
43. Evaluate int(a, t=3 .. 5), if a(t) = 5/(t^2) + 3/t.
44. Calculate int(y * cot(3*y^2) * csc(3*y^2), y).
45. If $f(x) = x^3 - x^2 - 6x$, find $\int f(x)dx.$
46. Evaluate the integral: $\int \frac{x}{x-6}dx$
47. Evaluate the integral: $\int \frac{x-9}{(x+5)(x-2)}dx$
48. Evaluate the integral: $\int_2^3 \frac{1}{x^2-1}dx$
49. Evaluate the integral: $\int_0^1 \frac{x-1}{x^2+3x+2}dx$

Answers for Sample Benchmark Questions

Note: I've tried to type and check these solutions carefully. Let me know if you think I've made an error.

Evaluating limits

1. If $y = 1/3x^2$, find the limit of y as $x \to \infty.$ 0 (or 0^+)
2. If $y = e^{-2x}$, find the limit of y as $x \to \infty.$ 0 (or 0^+)
3. If $y = e^{-2x}$, find the limit of y as $x \to -\infty.$ ∞ (or +∞)
4. If $y = 1/3x$, find the limit of y as $x \to 0^-$ -∞
5. If $y = 1/3x$, find the limit of y as $x \to 0^+$ +∞
6. If $y = 1/3x$, find the limit of y as $x \to 0$

   The limit is undefined because it approaches different values from the two different sides.
7. If $y = 1/5x^2$, find the limit of y as $x \to 0$

   ∞ (or +∞), because it approaches +∞ from both sides
8. If $y = e^{-2x}$, find the limit of y as $x \to 0$ 1
9. Calculate the limit: $\lim_{x\to\infty} \frac{5x^2+3x}{6-3x^2}.$ -5/3 (When we go to ∞ we consider just the highest power of x in the numerator and the highest power of x in the denominator.)

10. Calculate the limit: $\lim_{x\to 0}\frac{5x^2+3x}{6-3x^2}$. 0/6 = 0

11. Calculate the limit: $\lim_{x\to\infty} 5e^{-3x}$. 5 x 0 = 0

12. Calculate the limit: $\lim_{x\to 0} 5e^{-3x}$. 5 x 1 = 5

13. Calculate the limit: $\lim_{x\to 3}\frac{5x^2}{(x-3)(x+3)}$. Does the graph of this function have a vertical asymptote, a jump discontinuity, or a "hole" at the point where x = 3? Explain.
The limit is undefined because it approaches different values from the two different sides of 3. This function has a vertical asymptote at x = 3.

14. Calculate the limit: $\lim_{x\to 3}\frac{5x^2(x-3)}{x^2-9}$. Does the graph of this function have a vertical asymptote, a jump discontinuity, or a "hole" at the point where x = 3? Explain.

The limit as x → 3 is 7.5. The expression $\frac{5x^2(x-3)}{x^2-9}$ simplifies to $\frac{5x^2}{x+3}$, thus removing the "problem" at x = 3. The graph of this function has a "hole" at x = 3.

15. Calculate the limit: $\lim_{x\to 3}\frac{(x-3)(x+5)}{x^2-9}$. Does the graph of this function have a vertical asymptote, a jump discontinuity, or a "hole" at the point where x = 3? Explain.

The limit as x → 3 is 4/3. The expression $\frac{(x-3)(x+5)}{x^2-9}$ simplifies to $\frac{(x+5)}{(x+3)}$, thus removing the "problem" at x = 3. The graph of this function has a "hole" at x = 3.

Calculating derivatives

16. Find $f'(t)$ if $f(t)=3t^2+\sqrt{t}-5/t$. $6t+\frac{1}{2\sqrt{t}}+\frac{5}{t^2}$

17. Find the derivative of $x^2\cos(x)$. 2x cos(x) – x^2 sin(x)

18. Find the derivative of $R(t)=3t+e^{3t}-\pi$. 3 + 3 e^{3t}

19. Evaluate $f(x)=e^{-5x}$ at the point where x = 0. f(0) = 1

20. Evaluate the derivative of $f(x)=e^{-5x}$ at the point where x = 0. f'(0) = -5

21. Find the slope of the tangent line to the graph of $x^2\cos(x)$ at the point where $x=\pi/3$.

$$\frac{\pi}{3}-\frac{\pi^2\sqrt{3}}{18}$$

22. Find the equation of the tangent line to the graph of $x^2\cos(x)$ at the point where $x=\pi/3$.

$$y=m(x-x_1)+y_1=\left(\frac{\pi}{3}-\frac{\pi^2\sqrt{3}}{18}\right)\left(x-\frac{\pi}{3}\right)+\frac{\pi^2}{18}$$

23. If $g(x)=3x^2\exp(x)$, find dg/dx. $e^x(3x^2+6x)$

24. At what points (t, r(t)) will the derivative of $r(t) = 4t^3 + 3t^2$ be equal to 0? t=0 or t=-½

25. For what values of y will the derivative of $f(y) = 5y^3 - 3y^2 + 6y - 4$ be equal to 0?

$f'(y) = 15y^2 - 6y + 6.$ Since 36 – 4(15)(6) is less than 0, so that $\sqrt{36 - 4(15)(6)}$ is imaginary. Thus, the derivative of f(y) cannot equal 0 at any real number y.

26. Find the slope of the tangent line to the graph of $f(t) = 3t^2 - 5t$ at the point where *t = -1*.

The slope of the tangent line at t = -1 is given by f'(-1) = -11.

27. Find the slope of the tangent line to the graph of $f(x) = \dfrac{1}{3x}$ at the point where *x = 5*.

The slope of the tangent line at x = 5 is given by f'(5) = -1 / 75.

28. If $f(x) = x^3 - x^2 - 6x$, find $\dfrac{df}{dx}$. $3x^2 - 2x - 6$

29. Find the derivative of $f(x) = \dfrac{x^2}{\sin(3x)}$. $\dfrac{2x\sin(3x) - x^2\cos(3x)}{\sin^2(3x)}$

30. Evaluate the derivative of $r(\theta) = \dfrac{\sin(\theta)}{(\theta^2 + 3)}$ at the point where θ = 0. r'(0) = 1/3

Calculating antiderivatives and definite integrals

31. Find the antiderivative of $y(x) = 5x^2 + \pi x + 3$. $\dfrac{5}{3}x^3 + \dfrac{\pi}{2}x^2 + 3x + C$

32. If $y(x) = 5x^2 + \pi x + 3$, evaluate $\int_{-2}^{0} y(x)dx.$ $\dfrac{58}{3} + 2\pi$

33. Find the antiderivative of $R(t) = 3t + e^{3t} - \pi$. $\dfrac{3}{2}t^2 + \dfrac{1}{3}e^{3t} - \pi t + C$

34. Evaluate the definite integral: $\int_{1}^{5} 5x^2 + \pi x + 3.$ $\left(\dfrac{5^4}{3} + \dfrac{25\pi}{2} + 15\right) - \left(\dfrac{5}{3} + \dfrac{\pi}{2} + 3\right)$

35. Find the antiderivative: $f(x) = x\cos(x).$ cos(x) + x sin(x) + C

36. Evaluate $\int_{0}^{\pi} x\cos(x)dx.$ -2

37. Find the antiderivative: $g(y) = y\cos(y^2).$ $\dfrac{1}{2}\sin(y^2) + C$

38. Evaluate $\int_{0}^{\pi/2} y\sin(y^2)dy$ $\dfrac{1}{2}\sin\left(\dfrac{\pi^2}{4}\right) - 0$ (remember that sin(0) = 0)

39. Find the antiderivative: $\int \dfrac{\sec^2(4x)}{\tan(4x)}dx = \int \dfrac{(\sec(4x))^2 dx}{\tan(4x)}.$ $\dfrac{1}{4}\ln(\tan(4x)) + C$

40. Find int(y, x), if y(x) = 5*x^2 + sin(x) + 3. (5/3)*x^3 – cos(x) + 3x

41. Evaluate int(y, x=-Pi..Pi), if y(x) = 5*x^2 + sin(x) + 3.

((5/3)*Pi^3 – (-1) + 3*Pi) – ((5/3)*(-Pi)^3 – (-1) + 3*(-Pi))
= (5/3)*Pi^3 + 1 + 3*Pi + (5/3)*Pi^3 – 1 + 3*Pi
= (10/3)*Pi^3 + 6*Pi

42. Find int(a, t), if a(t) = 5/(t^2) + 3/t. -5/t + 3*ln(|t|) + C
43. Evaluate int(a, t=3 .. 5), if a(t) = 5/(t^2) + 3/t. 2/3 – 3*ln(3) + 3*ln(5)
44. Calculate int(y * cot(3*y^2) * csc(3*y^2), y). -1/6 * csc(3*y^2) + C
45. If $f(x) = x^3 - x^2 - 6x$, find $\int f(x)dx.$ (x^4)/4 – (x^3)/3 – 3x^2 + C

The next three problems can all be done using a partial fraction decomposition.

46. Evaluate the integral: $\int \frac{x}{x-6}dx$ x + x * ln (|x - 6|) + C
47. Evaluate the integral: $\int \frac{x-9}{(x+5)(x-2)}dx$ 2*ln(|x+5|) – ln(|x-2|) + C
48. Evaluate the integral: $\int_2^3 \frac{1}{x^2-1}dx$ (1/2)*ln(3/2)
49. Evaluate the integral: $\int_0^1 \frac{x-1}{x^2+3x+2}dx$ -5 ln(2) + 3 ln(3)